SEQUENTIAL CONTESTS WHEN THE PRECISION OF OBSERVATION IS ENDOGENOUS

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ABSTRACT. This paper explores a sequential contest where the second-mover can, at some cost, invests in a more or less precise signal on the leader's action before making his own move. We show that this signaling structure a la Weibull et al. (2007) guarantees the existence of a mixed-strategy perfect Bayesian equilibrium that fully preserves the value of commitment. We also prove that the profits of the first-mover in this type of equilibrium are always higher than in the subgame perfect equilibrium of the standard sequential contest.

1. INTRODUCTION

There is a sizable literature on leader-follower games initiated by Bagwell (1995) in which the lack of robustness of the value of commitment[1] runs like a leitmotif. The literature analyzing the fragility of commitment in sequential contest games is much smaller. Dixit (1987), analyzes the conditions under which a player benefits from moving first rather than playing in a simultaneous contest.[2] A key result here is that the profits of the first-mover are always higher with respect to his profits in the Nash equilibrium of the simultaneous contest. Following Dixit, most of the relevant literature[3] assumes that the follower can *perfectly* and *costlessly* observe the actual effort of the leader. A notable exception is Morgan and Várdy (2007) who extend the standard sequential contest to a situation where the second player must pay a small cost in order to *perfectly* observe the first player's move. While this is a good approximation in a variety of contests, the information pattern for the second-mover is often more complex. There are in fact many contests where the second-mover can only observe a more or less precise signal about the actual move of the leader; instead of observing her actual effort, the follower can typically increase the precision of his signal by incurring some disutility, which depends on the size of the precision. Examples abound:

–Litigation: There, the plaintiff and the defendant compete to influence the verdict of the Court in their favour. In general, a trial will take place only if the plaintiff has developed sufficient evidence. The defendant may therefore run a pre-trial investigation by hiring the advice of a public relations firm or a team of investigators in order to better jam the arguments of the plaintiff's lawyer from reaching the judge if the case is brought before a Court.

– Political competition: Consider an incumbent politician who intends to run for her re-election. Each candidate puts in effort in order to influence the voters through the production of arguments in favour of their policy orientation. Citizens know the policies offered by the candidate in power (i.e., the follower), while the front-runner in opposition (i.e., the leader) may deliberately let some uncertainty about his exact policy's position with regard to some particular issues. Thus, the politician in power may gather some information about the challenger's actual orientation in order to derail the debate in her favour during the campaign.

–Interest groups: Think of two interest groups, which compete over the approval of a policy. There, the rival interest group who seeks the rejection of the policy (the follower) can hire a lobbying firm to find out about a publication, a gossip exchanged or any event before it is widely known, in order to seek out potentially explosive information that might be hidden away by the other interest group before the Parliament or Congress has voted.

This paper aims at capturing these situations. We do so by analyzing a sequential contest model where the follower may, at some cost or disutility of effort, choose to increase the precision of a noisy signal about the leader's move. It turns out that the problem faced by the follower is similar to the decision-theoretic analysis of Weibull et al.(2007). Hence, from a theoretical point of view, one of the main innovation of this paper is to analyze such a class of decision problems in a strategic environment.

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[1] In this paper, we use the terms "value of commitment" and "first-mover advantage" interchangeably. Both terms refer to the extra equilibrium payoff the leader gets from moving first, as compared to his equilibrium payoff when the players move simultaneously.

[2] See also Baye and Shin (1999) for additional comments on Dixit (1987).

[3] Baik and Shogren (1992) and Leininger (1993) have confirmed the relevance of the sequential contest model. They study contests where the order of moves is endogenously determined by the contestants and conclude that a sequential contest game arises in equilibrium.

Our results show that when the first-mover is more likely to win in a simultaneous contest, i.e., he is the *favorite*, and the second-mover is the *underdog* [the terms come from Dixit, 1987], then the leader plays in a mixed-strategy perfect Bayesian equilibrium and the value of commitment is preserved completely. In fact, we show somewhat more than this, since we also demonstrate that the leader always obtains higher profits in such perfect Bayesian equilibria than in the unique subgame perfect Nash equilibrium of the standard sequential contest. Therefore, our results provide an ordinal characterization of the value of commitment in sequential contests.

These results are in stark contrast with Morgan and Várdy (2007) (MV hereafter), who demonstrate that if the follower faces a cost for observing the leader's action perfectly, then the value of commitment in sequential contests is *completely undermined*. From this perspective our results show that there is still a value of commitment when one considers the richer family of noisy sequential contests. What formally accounts for this difference? In short, this is because MV implicitly assume that the technology of signal is binary. Intuitively, in a sequential contest with noisy signals, the first-mover may benefit from the fact that his own move is imperfectly revealed to the follower. By contrast, all the perfect Bayesian Nash equilibria wherein the leader plays a pure action coincide with the pure Nash equilibria of the simultaneous contest. The argument behind this result is thus simple. In a perfect Bayesian equilibrium, the follower's beliefs must be correct. Hence, in a pure-strategy perfect Bayesian equilibrium it cannot be optimal for the follower to disburse any amount of money to merely confirm what he already knows. The upshot is thus that when there are some sufficiently small costs of precision, the second-mover finds valuable to observe an imperfect (but informative) signal if the behavior of the leader is unpredictable. Hence, in equilibria in which the leader plays a non-degenerate mixed strategy, the profits of the leader in the induced noisy sequential contest are higher than in the equilibria in pure strategies. This restores the value of commitment.

Alternatively put, this result proves that the value of commitment is lost or (potentially) preserved depending on the coarseness of the signaling structure: if the technology of signal is sufficiently fine relative to the cost of observation i.e., there is a non-binary set of potential signals, then the value of commitment is preserved; if the structure is binary—when the follower pays he perfectly observes 1's action, otherwise the follower does not observe anything—,then the value of commitment is lost.

Relationship with the literature

To the best of our knowledge, the idea of incorporating a cost for more precise observations for the second mover in a sequential game is new. However, our work is related to the information-economics literature developed initially by Kihlstrom (1974a,b) and Radner and Stiglitz (1984). More precisely, our model follows the optimal acquisition of "instrumental" information initially developed by Kihlstrom (1974a,b). In this approach, the information is instrumental in the sense that it influences utility only indirectly, via the decision- making process. In particular, Kihlstrom (1974) formulates a demand theory for instrumental information by incorporating the cost for more precise observations in the budget constraint of the consumer.

At a broad level, the present noisy sequential contest model is also related to the seminal contribution of Bagwell (1995), which studies the value of commitment in the context of an oligopoly setting where the imperfection of information of the follower is exogenous and costless. Hence, the present model differs from this literature in one crucial way since in our setup the noise contaminating the signaling technology is endogenously determined by the follower at a cost depending on his choice of a precision of observation.

The remainder of the paper is organized as follows. The next section presents the contest model. Section 3 presents the basic sequential contest, and our generalized version of this model. Section 4 presents our main result and Section 5 concludes. For the sake of exposition all the proofs are given in the Appendix.

2. MODEL

There are two risk neutral players 1 and 2 who participate in a contest. Players attribute a value to the object (or prize) that is allocated in the contest. Let V_i denote the (positive and finite) value of the prize to player i. Both players know their own value and the value their opponent attributes to winning the contest. Let $P(x_1, x_2)$ be a continuously differentiable contest success function which gives the probability that the object will be awarded to player 1 when effort $x_i \in \mathbb{R}_+$, $i = 1,2$, are expended. $C_i(x_i)$ denotes the cost to player i of expending effort x_i. Hence, player 1's expected payoff is,

$$\pi_1(x_1, x_2) = P(x_1, x_2)V_1 - C_1(x_1), \tag{2.1}$$

and player 2's payoff is,

$$\pi_2(x_1, x_2) = [1 - P(x_1, x_2)]V_2 - C_2(x_2). \tag{2.2}$$

Following much of the contest literature, we assume the following class of contest success functions: $P_1(x_1,x_2) \equiv \frac{\partial P(x_1,x_2)}{\partial x_1} > 0$, $P_2(x_1,x_2) \equiv \frac{\partial P(x_1,x_2)}{\partial x_2} < 0$ for all x_1, x_2. That is, player 1's chances of winning are increasing in his own effort and decreasing in his rival's effort. This class of success functions encompasses the Logit form

$$P(x_1,x_2) = \begin{cases} \frac{f_1(x_1)}{f_1(x_1)+f_2(x_2)} \text{ if } (x_1,x_2) \neq (0,0); \\ \frac{1}{2} \text{ otherwise,} \end{cases}$$

where $f_i(\cdot)$ measures the impact of x_i in the contest with $f_i(0) = 0$, $f_i^{'}(\cdot) > 0$ and $f_i^{''}(\cdot) \leqslant 0$ i.e., $f_i(\cdot)$ is strictly increasing and concave.[4] Clark and Riis (1998) and Skaperdas (1996) offer axiomatic characterizations for these contest success functions and Corchón and Dahm (2009) derive microeconomic foundations for this class of win probabilities.

We also assume that C_i is continuously differentiable, strictly increasing, and (weakly) convex and that $\lim_{x_i \to \infty} C_i(x_i) = \infty$.

Simultaneous contests. As a starting point, we consider the case where x_1 and x_2 are selected simultaneously. The explicit conditions that guarantee the existence of a pure strategy Nash equilibrium in the case of general payoff functions are given in Appendix. If the contest success function is of the Logit form and the cost of effort is, $C_i(x_i) = x_i$, the following general facts are well-known in the contest literature (see e.g. Yildirim, 2005). For the reader's convenience, we reproduce them below.

Fact 1. Player i's problem is strictly concave in x_i.

Fact 1 implies that for each x_j, player i has a unique best response $x_i(x_j)$ satisfying the first-order condition for player i, $\frac{\partial \pi_i}{\partial x_i} = 0$. Together with the earlier assumptions made on P and C_i, this implies that $x_i(x_j)$ is continuously differentiable and bounded.

Fact 2. The best-response function $x_i(x_j)$ is strictly increasing when efforts x_i and x_j are such that $f_i(x_i) > f_j(x_j)$, reaches its maximum when $f_i(x_i) = f_j(x_j)$, and is strictly decreasing when $f_i(x_i) < f_j(x_j)$.

Fact 2 implies that the best-response functions $x_1(x_2)$ and $x_2(x_1)$ cross the locus where $f_i(x_i) = f_j(x_j)$ exactly once. Therefore, there exists a unique Nash equilibrium (x_1^*, x_2^*) if the contest success function takes the Logit form and if the cost of effort is equal to the effort itself. In the case of general contest success functions and cost functions, the existence of a pure strategy Nash equilibrium is also guaranteed by extending suitably the regularity conditions (see Appendix A.1).

3. Sequential contests

Next, suppose that the contest is played sequentially. That is, player 1 chooses x_1, player 2 *costlessly* and *perfectly* observes x_1, and then chooses x_2. Thus, the main difference with the simultaneous contest concerns player 1's optimization problem: player 1 is to choose x_1 to maximize $\pi_1(x_1, x_2(x_1))$, recognizing that player 2 will be playing a best response to x_1. In this sequential set-up, the following fact holds for general contest success functions and cost of effort, provided C_i is sufficiently convex.

Fact 3. Given that player 2 is playing a best response, player 1's problem is strictly concave in x_1.

Together with the additional regularity condition that the players' payoff functions are strictly concave in the simultaneous contest (see Appendix A.1), Fact 3 imply that there exists a unique subgame perfect Nash equilibrium, $(x_1^{**}, x_2(x_1^{**}))$, in the sequential contest without observation costs.

Following Dixit (1987), we say that there is a *value to commitment* if the profits of the first mover in the subgame perfect Nash equilibrium (SPNE) of the sequential contest are higher than in any pure strategy Nash equilibrium (NE) of the simultaneous contest. In the case of the Logit form success function, this amounts to imposing $f_1(x_1^*) \neq f_2(x_2^*)$ where (x_1^*, x_2^*) denotes the Nash equilibrium profile of the simultaneous contest.

For general payoff functions, fix the effort levels of the two players at one of the pure strategy Nash equilibrium, (x_1^*, x_2^*), of the simultaneous contest. Then, it is readily shown that the condition[5]

$$V_1 \left[P_2(x_1^*, x_2(x_1^*)) \frac{\partial x_2(x_1^*)}{\partial x_1} \right] \neq 0,$$

with $x_2(x_1^*) = x_2^*$, guarantees that there is positive value of commitment. In the remainder of this paper we assume that this condition holds. Together, the assumptions that players' payoff functions are strictly concave in the simultaneous case and player 1's payoff function is strictly concave in the sequential contest, imply that we leave

[4] This class of contest success functions was initially proposed by Tullock (1980).

[5] see e.g. Morgan and Várdy (2007, Appendix A.3, p.337).

outside our study the class of sequential contests where the first mover's problem is ill-behaved, fail to have an equilibrium in pure strategies, or the follower is non-reactive.

3.1. **Sequential contests with endogenous precision of observation.** We first provide an informal description of the time line of our game.

Stage 1: Player 1 chooses a strategy $\sigma_1 \in \Delta_1$. Here, the choice of a mixed strategy can be thought of as the choice of a randomization device wherein player 1 delegates her choice of an effort to a trustworthy party while the choice of a degenerate randomization device can be interpreted as a situation where player 1 picks herself an effort level $x_1 \in X_1$. The decision of delegating or not delegating as well as the randomization device (if any) is not observed by player 2.

Stage 2: The outcome of the randomization device is realized or the pure choice of player 1 is made. Whether there is delegation or not, the ultimate realization or choice x_1 is imperfectly observed by player 2: By default, player 2 observes an arbitrary imperfect signal about the outcome of the randomization device or about the actual choice made by 1. Player 2 picks an effort level $x_2 \in X_2$ and may choose to increase the precision $\tau \in [0,1]$ of these signals. More specifically, Stage 2 is described as follows.

Endogenous imperfect observation. Assume that the follower can, at some cost, invests in a more or less reliable source of information on the leader's action before making his own move, e.g. a firm can invest in some competitive intelligence activities[6]. Formally, before deciding on x_2, player 2 can increase the "precision" of his or her observation of player 1's choice of effort, x_1, at a continuously varying intensity $\tau \in [0,1]$. The parameter τ can be thought of as the *efficacy* or *type* of the source of information. A 0-type intelligence service gives no information, while a 1-type intelligence service perfectly reveals the actual choice of the leader. For any other type of information sources, $\tau \in (0,1)$, the leader's actual choice will be generally imperfectly observed. Alternatively, τ can be seen as the amount of costly attention the follower pays to a perfectly informative source i.e., he endogenously chooses how carefully to listen to it.

Signaling technology. For the remainder of the paper, in order to simplify exposition, we shall suppose that player 1 chooses effort according to a mixed strategy, σ_1, whose support is a *finite* subset of $\mathbb{R}_+$. For the purpose of our analysis, this is without loss of generality. [7] In Appendix A.3 we show how our signaling structure extends to the case of a mixed strategies with infinite support.

Let a_1 denote the *actual effort* of player 1 when he plays a mixed strategy σ_1 i.e. $a_1 = x_1$ if x_1 is the outcome of the mixed strategy, σ_1. When the follower invests in a τ-type source of information, with a precision $\tau > 0$, the likelihood of player 2 of receiving the signal $\phi = x_1$ conditional to $a_1 = x_1$ is given by

$$\Pr^{\tau}\{\phi = x_1 \mid x_1\} = \tau + (1-\tau)\mu_2(x_1) \tag{3.1}$$

with,

$$\mu_2(x_1) \equiv \begin{cases} \sigma_1(x_1) \text{ if 1 has chosen the non-degenerate randomization device } \sigma_1; \\ \mu_2^o(x_1) \text{ if 1 has chosen a pure action } x_1. \end{cases}$$

where $\mu_2 \equiv \sigma_1$ is an *independent* probability distribution and μ_2^o an arbitrary probability distribution.

Several remarks are in order.

First, let us stress that μ_2^o can be any arbitrary distribution. An immediate implication is thus that our setup encompasses the case where the observability of the follower is *independent* of whether player 1 plays a pure or a mixed strategy. Alternatively put, for an equal level of precision $\tau > 0$, it is always possible to choose μ_2^o such that the observability of the follower does not depend on whether the actual action a_1 is the result of a mixed or pure strategy.

Also, notice that Eq.(3) is only defined for $\tau > 0$. If $\tau = 0$, then player 2 obtains no information about 1's choice, as in a simultaneous contest. Moreover, let us stress, once again, that the actual choice of the first-mover, a_1 (drawn from 1's randomized strategic action, σ_1), and the signal drawn from distribution, $\mu_2 \equiv \sigma_1$, are *independent events*.

It also follows from Eq.(3) that the parameter τ represents a proper measure of the precision of observation. When the first mover plays a non-degenerate mixed strategy and player 2 chooses a precision level, $0 < \tau < 1$, the signal received by player 2 about the first mover's choice of action is garbled. However, if player 2 chooses $\tau = 1$ the observation is perfect i.e. the signal received by the follower is perfectly correlated to 1's actual move. In general,

[6]Competitive intelligence is the name given to the systematic and ethical approach for gathering, analyzing and managing information that can affect a firm's plans, decisions and operations.

[7]In fact, it is enough to establish the existence of a mixed strategy with finite support that strictly dominates all the pure actions of the first-mover to argue that there is value to commitment.

when player 2 chooses a high level of precision, he increases his chance to observe a signal corresponding to 1's actual choice.
To summarize, the distribution of signals conditional to $a_1 = x_1$ can be thought of as the following two-stage lottery:

1. The randomization devices selects action x_1 according to 1's non-degenerate mixed strategy σ_1 or player 1 picks a pure action x_1.
2. When player 1 uses a randomization device σ_1: With probability τ, player 2 observes the actual effort, $a_1 = x_1$, while with probability $1-\tau$ he is reported a signal in the support of μ_2. This signal is drawn according to the (independent) probability distribution $\mu_2 \equiv \sigma_1$. The follower cannot distinguish between the received signals that represent the actual move of the leader and signals that were drawn from the independent distribution μ_2. When player 1 plays a pure strategy, the same lottery occurs with an arbitrary default distribution μ_2^o.

Disutility of precision. Associated with each source of information of type, τ, is a cost or disutility $c(\tau)$ to the second mover. We will consider the class of functions $c : [0,1] \to \mathbb{R}_+$ which are continuously differentiable with $c(0) = 0$ and $\lim_{\tau \to 1} c(\tau) = +\infty$. Moreover, we consider disutility functions that are strictly increasing i.e. $c'(\tau) > 0$ for all $\tau \in [0,1)$. When we interpret τ as the efficacy of the intelligence service of the follower, these assumptions say that the follower's marginal cost of improving the efficacy of his intelligence service is increasing, and that the cost of having an intelligence service that always reports the actual state of the world is prohibitive. Thereafter $\mathscr{C}$ denote the class of cost functions satisfying all the properties mentioned above.
Payoffs. In this setting, player 1's total (expected) payoff is the same as in the standard contest i.e. $V_1 P(x_1, x_2) - C_1(x_1) \equiv \pi_1(x_1, x_2)$. Similarly, let $V_2[1 - P(x_1, x_2)] - C_2(x_2) \equiv \pi_2(x_1, x_2)$. Then, the total (expected) utility from player 2 investing τ and exerting effort level x_2 when the first mover plays x_1 becomes,

$$\pi_2(x_1, x_2) - \delta c(\tau) \equiv \pi_2(x_1, x_2, \tau),$$

where δ is a positive scalar that represents the relative weight the second mover attaches to the disutility or cost of precision of observation. Thereafter, we denote the resulting sequential game by Γ^δ.
Let us briefly discuss how our model is related to the standard sequential contest where player 2 costlessly and perfectly observes x_1. The "disutility weight" δ measures the importance of the cost or disutility associated with "precision" τ relative to the payoffs in the underlying game. The limiting case $\delta = 0$ represents a situation in which player 2 is able to perfectly observe 1's action as if he played the sequential contest without observation costs. Thus, when $\delta = 0$ our model collapses to the standard sequential contest *without* observation costs.
Timing. The timing of Γ^δ is summarized as follows:

1. Player 1 chooses a strategy and player 2 a precision level τ.
2. If 1 chooses a randomization device σ_1 an outcome a_1 of the mixed strategy σ_1 is realized and with a probability $(1-\tau)$ a chance move draws a noisy signal from an independent distribution $\mu_2 \equiv \sigma_1$. Otherwise, a similar chance move occurs with an arbitrary distribution μ_2^o.
3. Prior to deciding on x_2, player 2 observes an imperfect signal if $0 < \tau < 1$. If $\tau = 0$, player 2 obtains no signal.
4. The game ends and the players receive their payoffs.

Strategies and Equilibria in Γ^δ. We will analyze the extensive form game of imperfect information Γ^δ for its (weak) perfect Bayesian equilibria. In a perfect Bayesian equilibrium (henceforth PBE) strategies are optimal given players' beliefs and beliefs are derived from Bayes' rule whenever possible i.e., there is weak consistency of beliefs with strategies.
Let Δ_1 denote the set of all discrete probability distributions over $\mathbb{R}_+$.[8] Formally, (σ_1, β_1) is a strategy-belief pair for player 1 (the first mover) in Γ^δ with σ_1 a a mixed action $\sigma_1 \in \Delta_1$ and $\beta_1 \in \Delta_2$. For player 2, a strategy-belief pair in Γ^δ is a pair (s_2, β_2), defined by:

1. a strategy s_2 is a pair of mappings, $(\hat{\tau}, \hat{x}_2) \equiv s_2$, where $\hat{\tau}$ is defined as $\hat{\tau} : \Delta_1 \ni \beta_2 \mapsto \hat{\tau}(\beta_2) \in [0,1]$ and $\hat{x}_2$ is the map $\hat{x}_2 : [0,1] \times X_1 \ni (\hat{\tau}(\beta_2), \phi) \mapsto x_2(\hat{\tau}(\beta_2), \phi) \in X_2$;
2. given a precision level, $\hat{\tau}(\beta_2) = \tau$, a system of beliefs $\hat{\beta}_2$ is defined as

$$\hat{\beta}_2(a_1, \phi, \tau) = \text{Prob}(a_1 | \phi, \tau)\beta_2(a_1) \text{ with } \sum_{(a_1,\phi)} \hat{\beta}_2(a_1, \phi, \tau) = 1.$$

[8] As already mentioned, the focus on these strategies is without loss of generality.

Following Yildirim (2005), we assume that the leader is the favorite and the follower is the underdog i.e., $f_1(x_1) > f_2(x_2)$, and we will say – following Dixit (1987) –, that there is *value to commitment* in Γ^δ if the profits of the first mover in a perfect Bayesian equilibrium of Γ^δ are higher than in any pure strategy Nash equilibrium of the simultaneous contest.

4. MAIN RESULT

We now present the main result of the paper: there is value to commitment when the precision of observation is costly. For some sufficiently small costs of precision with $c \in \mathscr{C}$ there is value to commitment in Γ^δ in any perfect Bayesian equilibrium in mixed strategy for player 1. In particular, the profits of the leader are higher in Γ^δ than in the (unique) SPE of the standard sequential contest. **Proof** see Appendix. ■

Pending, the details of the proof in the Appendix, we here sketch and discuss the main ideas of the proof.

First, if we focus attention to strategies wherein player 1 does not resort to a randomization device i.e., he plays a pure action, we have the following result.

Claim 0 *(see Claim 0, Appendix A.2) When we restrict the analysis to pure strategies the set of pure-strategy PBEs of the sequential contest with costly observation precision coincides with the set of pure NEs of the simultaneous contest without cost of observation (e.g. the unique NE in case of a Logit contest success function).*

The intuition behind this result is simple. Player 2 will never pay for a $\tau \in (0,1)$ in order to get a (noisy) signal about the strategy of player 1 (he gets no signal if $\tau = 0$), since in a pure equilibrium he already knows the actual state of affair (2's belief is consistent with player 1's action). The upshot is thus that in a PBE in pure strategies, players get the same equilibrium payoffs as in the original simultaneous contest.

Second, we turn to mixed strategies.

Lemma 0 *In Γ^δ, there always exists a mixed strategy i.e., a randomization device, that strictly dominates the pure actions of player* 1.

Intuitively, this result follows from the fact that if player 1 is playing a non-degenerate mixed strategy, player 2's best response varies depending on the realization of player 1's actual effort. This means that, for sufficiently small costs of observation, player 2 will indeed find it optimal to pay in order to observe an imperfect (but informative) signal about the leader's effort. By contrast, when 1 plays a pure action, this allows player 2 to perfectly predict the leader's equilibrium action. This in turn makes player 2 choose a 0-type source of information. As a result, the first-mover always prefer inducing a noisy sequential contest by rendering his behavior unpredictable. And indeed, in these equilibria, player 1 gets higher profits than in the pure-strategy PBEs which by Claim 0 coincide with the equilibria of the simultaneous contest.

Finally, note that in this setting, the use of mixed-strategies can thus be motivated analogously to the classical argument in favour of mixed-strategies, namely that mixed-strategies produce unpredictable choices that cannot be exploited by an opponent. In the context of a noisy sequential contest, this permits the leader to oblige the follower to pay for an imperfect signal about the leader's actual effort which restores his value to commitment.

5. CONCLUDING REMARKS

We have analyzed a sequential contest where the second mover can acquire a better quality of signal of the actual move of the leader at some monetary costs. Our main innovation is to incorporate such a class of decision problems a la Weibull et al. (2007) in a strategic environment. In this setting, we demonstrate the existence of a mixed-strategy perfect Bayesian equilibrium that fully preserves the value of commitment. Our analysis also provides an ordinal ranking on the value of commitment with respect to the equilibria of the standard sequential and simultaneous contests.

In this paper we have focused on contests. Our setup can be applied to many games where the first mover's problem is strictly concave as the Cournot/Stackelberg quantity competition. We leave this to future research.

Appendix.

A.1 Simultaneous contest.

Suppose that the two players compete simultaneously. To ensure that this problem is well-behaved we make the following regularity assumption.

Assumption 1 Player i's problem is strictly concave. That is, for all x_1, x_2 :

$$\mathrm{P}_{11}(x_1,x_2)\mathrm{V}_1 - \mathrm{C}_1''(x_1) \leqslant 0$$
$$\text{and } -\mathrm{P}_{22}(x_1,x_2)\mathrm{V}_2 - \mathrm{C}_2''(x_2) \leqslant 0.$$

A.1 is verified when C_i sufficiently convex or $\mathrm{P}_{11}(x_1,x_2) \equiv \frac{\partial^2 \mathrm{P}(x_1,x_2)}{\partial x_1^2} < 0$ and $\mathrm{P}_{22}(x_1,x_2) \equiv \frac{\partial^2 \mathrm{P}(x_1,x_2)}{\partial x_2^2} > 0$ for all x_1, x_2. This guarantees that the first-order conditions are both necessary and sufficient for characterizing the best-response functions of the players.

The main conclusion we derive from these observations is that there exists a pure strategy Nash equilibrium (x_1^*, x_2^*) satisfying

$$\mathrm{P}_1(x_1^*,x_2^*)\mathrm{V}_1 - \mathrm{C}_1'(x_1^*) = 0$$
$$\text{and } -\mathrm{P}_2(x_1^*,x_2^*)\mathrm{V}_2 - \mathrm{C}_2'(x_2^*) = 0.$$

A.2 Sequential contest with costly observation precision

Next, suppose that the contest is played sequentially as follows. Player 1 chooses x_1 and player 2 at a cost chooses a precision level τ. Then, player 2 observes a noisy signal about player 1's actual effort that is parametrized by τ and chooses x_2.

The two-stage decision problem of player 2. Before observing a signal, the second-mover can improve the precision of his signal by purchasing a precision of observation τ.

Updating signals. At the last stage, given a level of precision τ and on observing the signal, $\phi \in \mathrm{supp}(\sigma_1)$, player 2 computes, using Bayes rule, that the event a_1 occurs with probability $\Pr^\tau\{a_1 \mid \phi\}$. Formally, let $\mathrm{supp}(\sigma_1)$ denote the (finite) support of the mixed strategy σ_1 of player 1. Then, if player 2 observes a signal ϕ then Bayes' rule implies that player 2 must believe that $a_1 = x_1$ with probability[9]

$$\beta_2(a_1 = x_1 \mid \phi = x_1; \tau) = \frac{\tau\sigma_1(x_1) + (1-\tau)\sigma_1(x_1)^2}{\tau\sigma_1(x_1) + (1-\tau)\sigma_1(x_1)^2 + (1-\sigma_1(x_1))(1-\tau)\sigma_1(x_1)},$$

where $\sigma_1(x_1) \equiv \mu_2(x_1)$. Under a precision level $\tau > 0$ in game Γ^δ, after having observed a noisy signal ϕ, player 2 chooses the (unique)[10] action that maximizes his conditional expected payoff,

$$\textstyle\sum_{a_1 \in \mathrm{supp}(\sigma_1)} \pi_2(a_1,x_2)\beta_2(a_1 \mid \phi;\tau) - c(\tau) \equiv \pi_2(x_2,\tau \mid \phi).$$

Hence, conditionally on a pair, (τ,ϕ), with $\hat{\tau}(\sigma_1) = \tau$, the optimal choice of effort of player 2 is,

$$x_2(\tau,\phi) = \arg\max_{x_2'} \pi_2(x_2',\tau \mid \phi).$$

Let $\pi_2(a_1, x_2(\tau,\cdot),\tau)$ denote the maximal expected payoff of player 2 when player 1's actual choice is a_1 and player 2 received a signal ϕ with precision τ. With these notations, a precision parameter, $\hat{\tau}(\sigma_1) = \tau^*$, is optimal if,

$$\tau^* = \arg\max_{\tau \in [0,1]} \textstyle\sum_{(a_1,\phi) \in \mathrm{supp}(\sigma_1) \times \mathrm{supp}(\sigma_1)} \beta_2(a_1,\phi;\tau)\pi_2(a_1,x_2(\tau,\phi),\tau).$$

In the sequel, let $\mathbb{E}\pi_2(x_2(\tau,\cdot),\tau) \equiv \sum_{(a_1,\phi)\in \mathrm{supp}(\sigma_1)\times\mathrm{supp}(\sigma_1)} \beta_2(a_1,\phi;\tau)\pi_2(a_1,x_2(\tau,\phi),\tau)$ denote the maximand of this problem. There exists an optimal precision τ^*.

Proof. A precision parameter value τ is optimal if it solves

$$\max_{\tau[0,1]} \mathbb{E}\pi_2(x_2(\tau,\cdot),\tau)$$

We first observe that the maximand is continuous in τ. To see this, note that Berge's Theorem ensures that the maximizer, $x_2(\tau,\cdot)$, is continuous in τ. This follows as $\beta_2(a_1 \mid \phi;\tau)$ and $c \in \mathscr{C}$ are both continuous in τ. By assumption, the class of winning probabilities and the cost of effort of the follower, $\mathrm{C}_2(x_2)$, are assumed to be differentiable in x_2. Hence, they are also continuous in x_2. Together with the fact that $\beta_2(a_1,\phi;\tau) = \beta_2(\phi \mid a_1;\tau)\sigma_1(a_1)$ is also continuous in τ, this implies that the maximand is continuous in τ. The maximand is continuous in the choice variable τ, and this variable is constrained to a compact set, so the decision problem has a non-empty compact

[9]Recall that in our setup, the follower cannot observe a signal not contained in the support of the mixed action.

[10]That player 2 has a unique best response follows from Fact 1.

solution set, by Weierstrass' Maximum theorem. This completes the proof. ■

The following Lemma shows that player 2 will always choose an interior precision level, $0 < \tau^* < 1$, whenever player 1 plays a non-degenerate mixed strategy. As a consequence, it then follows that player 1 can always induce a noisy sequential contest if he plays a non-degenerate mixed strategy. When player 1 uses a non-degenerate mixed strategy, σ_1, the optimal precision level of player 2 in Γ^δ is such that $0 < \tau^* < 1$. **Proof.** First, notice that V_2 bounded, and $\lim_{\tau\to 1} c(\tau) = +\infty$, guarantee that player 2 always picks levels of precision $\tau \in [0,1)$, thereby proving that $\tau^* < 1$. Next, we show that if player 1 plays a non-degenerate mixed strategy, then the optimal level of precision in Γ^δ is $\tau^* > 0$, provided that there are some sufficiently small costs of precision. We first observe that when player 1 plays a mixed strategy, then the game becomes a simultaneous contest whenever player 1 does not pay to observe i.e. $\tau = 0$. Next, define $g : \tau \mapsto \mathbb{E}\pi_2(x_2(\tau,\cdot),\tau) - c(\tau)$. If $\tau = 0$, then in a PBE, the decision problem of player 2 boils down to,

$$\mathbb{E}\pi_2(x_2(0,\cdot),0) = \max_{x_2} \sum_{x_1} \sigma_1(x_1)\pi_2(x_1,x_2) \equiv g(0),$$

with $c(0) = 0$. This equation is automatically satisfied since, in equilibrium, beliefs are necessarily correct in the resulting simultaneous game. On the other hand, if $\tau = 1$, then player's 2 decision problem corresponds to the one of the standard sequential contest, $\max_{x_2} \pi_2(x_1,x_2)$, for every $x_1 \in \text{supp}(\sigma_1)$ in both frameworks. This yields the ex-ante optimal gross expected payoff,

$$\sum_{x_1 \in \text{supp}(\sigma_1)} \sigma_1(x_1) \max_{x_2} \pi_2(x_1,x_2) \equiv g(1).$$

g is a real-valued continuous function. Thus, by using the Intermediate Value Theorem, we can pick a precision $1 > \tau > 0$, such that $g(0) < g(\tau) < g(1)$. Hence, given a non-degenerate mixed strategy, σ_1, there exists a level of precision, $\widehat{\tau}(\sigma_1)$, such that $g(\widehat{\tau}(\sigma_1)) - g(0) > 0$. Together with the assumption $c(0) = 0$, this entails that $\widehat{\tau}(\sigma_1) = \tau^* > 0$ whenever the function $c \in \mathscr{C}$ is such that there exists a $\tau^* > 0$ with $g(\tau^*) - g(0) > c(\tau^*)$. Hence, in both frameworks, player 2 pays for a precision level $\tau^* > 0$ whenever the leader plays a non-degenerate mixed strategy, if $c \in \mathscr{C}$ is such that there exists a sufficiently small precision cost. This completes the proof. ■

The decision problem of player 1. Now we turn to player 1. Anticipating player 2's equilibrium precision level, τ^*, and a sequentially rational response to each observed signal ϕ, player 1 chooses a (mixed) strategy σ_1^*. We have the following claim concerning the equilibrium payoffs of player 1.

__**Claim**__ 0 *Let (x_1^*, x_2^*) be a NE (resp. the unique NE in case of a Logit contest success function) of the simultaneous contest without cost of observation. If we restrict the analysis of Γ^δ to pure strategies, then the PBEs strategies coincides with the pure equilibria (resp. the unique equilibrium) of the simultaneous contest*

Proof. The follower holds some beliefs about 1's action even without observing it. In equilibrium, these beliefs must be correct and, therefore, also degenerate. Degeneracy of correct beliefs implies that the follower can perfectly predict the leader's equilibrium action. In that case, it can never be optimal for the follower to improve his precision level τ, to merely confirm his (correct) beliefs. Therefore, in any pure-strategy PBE, the follower never observes the leader's action. Hence, in Γ^δ, when player 1 plays a pure strategy, player 2 chooses $\tau^* = 0$. By definition, this implies that players play in the simultaneous contest, thereby proving that players will play in the set of NE of the simultaneous contest. This completes the proof. ■

Notations: In the proofs below, we shall use the following notations. Let $\sigma_1 = (1-p,p)$ be an arbitrary mixed strategy with a two-point support, $\{\underline{x}_1, \overline{x}_1\} \equiv \text{supp}(\sigma_1)$, where player 1 plays $\underline{x}_1 < \overline{x}_1$ with $\sigma_1(\underline{x}_1) = (1-p)$ and $\overline{x}_1$ with $\sigma_1(\overline{x}_1) = p$.

__**Claim 1**__ *Let $(x_1^{**}, x_2(x_1^{**}))$ be the SPNE of the sequential contest without cost of observation. Suppose player 1 is the favorite i.e. $P_{12}(x_1^*, x_2^*) > 0$.*[11] *Then, there exists a PBE in Γ^δ wherein player 1 plays a mixed strategy $\sigma_1^\epsilon = (1-p,p)$, with $p \in (0,\overline{p}]$ and $\text{supp}(\sigma_1) = \{x_1^{**}, x_1^{**}+\epsilon\}$ with $\epsilon > 0$ that strictly dominates all his pure actions. Moreover, in this PBE the leader's payoffs are higher than in the unique SPE of the standard sequential contest.*

Proof. First, some definitions. Given a mixed strategy σ_1^ϵ, with support $\text{supp}(\sigma_1^\epsilon) = \{\underline{x}_1^{**}, x_1^{**}+\epsilon\}$, let $\mathrm{E}_{\beta_2^\tau(\cdot|\phi=x_1)}\widetilde{x}_1$ denote the induced *expected effort of player* 1 *from* 2*'s perspective when* 2 *observes a signal* $\phi = x_1$ *under a precision level* $\widehat{\tau}(\beta_2) = \tau$ with the requirement that $\beta_2 = \sigma_1$ in a PBE. Define

$$\sum_{\phi \in \text{supp}(\sigma_1^\epsilon)} \text{Prob}(\phi \mid x_1;\tau) \mathrm{P}(x_1, \widehat{x}_2(\mathrm{E}_{\beta_2^\tau(\cdot|\phi)}\widetilde{x}_1)) \equiv \widetilde{\mathrm{P}}(x_1, \widehat{x}_2),$$

[11] As shown in Dixit (1987), this amounts to the condition $f_1(x_1^*) > f_2(x_2^*)$ in the case of a Logit contest success function.

where $\mathrm{Prob}(\phi \mid x_1; \tau) = \beta_2(\phi \mid x_1; \tau)$. This is the conditional expected probability of winning of player 1 under the mixed strategy $\sigma_1^\epsilon = (1-p, p)$ when outcome $x_1 \in \mathrm{supp}(\sigma_1^\epsilon)$ is realized.
Lemma 2 implies that player 2 chooses an optimal precision level $1 > \tau^* > 0$ whenever player 1 plays a non-degenerate mixed strategy. Moreover, when 1 is the favorite, we have that $\widehat{x}_2$ is decreasing in x_1 (see Dixit, 1987). Under σ_1^ϵ with support $\mathrm{supp}(\sigma_1^\epsilon) = \{x_1^{**}, x_1^{**} + \epsilon\}$, notice that $\mathrm{E}_{\beta_2^\tau(\cdot|\phi)} \widetilde{x}_1 > x_1^{**}$ for any ϕ in the support of σ_1^ϵ. This follows since $\mathrm{E}_{\beta_2^\tau(\cdot|\phi)} \widetilde{x}_1$ is the convex combination (via the non-degenerate updated beliefs of player 2) of two effort levels $\underline{x}_1 = x_1^{**}$ and $\overline{x}_1(\epsilon) := x_1^{**} + \epsilon > x_1^{**}$. Hence, the condition $\mathrm{P}_12(x_1^*, x_2^*) > 0$ implies $\widehat{x}_2(\mathrm{E}_{\beta_2^\tau(\cdot|\phi)} \widetilde{x}_1) < \widehat{x}_2(x_1^{**})$ for any ϕ in the support of σ_1^ϵ by using the standard argument of Dixit (1987). Using the assumption, $\mathrm{P}_1 > 0$ and $\mathrm{P}_2 < 0$, we conclude that,

$$\mathrm{P}(x_1, \widehat{x}_2(\mathrm{E}_{\beta_2^\tau(\cdot|\phi)} \widetilde{x}_1)) > \mathrm{P}(x_1^{**}, \widehat{x}_2(x_1^{**})) \tag{5.1}$$

for all pair $(x_1, \phi) \in \mathrm{supp}(\sigma_1^\epsilon) \times \mathrm{supp}(\sigma_1^\epsilon)$.
Thus, we have that $\widetilde{\mathrm{P}}(\overline{x}_1(\epsilon), \widehat{x}_2) > \mathrm{P}(x_1^{**}, \widehat{x}_2(x_1^{**}))$ and $\widetilde{\mathrm{P}}(\underline{x}_1, \widehat{x}_2) > \mathrm{P}(x_1^{**}, \widehat{x}_2(x_1^{**}))$. By definition the total expected probability of winning of player 1 is a convex combination of $\widetilde{\mathrm{P}}(\overline{x}_1, \widehat{x}_2)$ and $\widetilde{\mathrm{P}}(\underline{x}_1, \widehat{x}_2)$, which implies that

$$p\widetilde{\mathrm{P}}(\overline{x}_1(\epsilon), \widehat{x}_2) + (1-p)\widetilde{\mathrm{P}}(\underline{x}_1, \widehat{x}_2) > \mathrm{P}(x_1^{**}, \widehat{x}_2(x_1^{**})), \tag{5.2}$$

for any $p \in (0,1)$. Given a non-degenerate mixture $(1-p, p)$, we note that the support of σ_1^ϵ implies that,

$$(1-p)\mathrm{C}_1(x_1^{**}) + p\mathrm{C}_1(\overline{x}_1(\epsilon)) > \mathrm{C}_1(x_1^{**}). \tag{5.3}$$

As a result, σ_1^ϵ strictly dominates x_1^{**} if

$$\mathrm{V}_1[p\widetilde{\mathrm{P}}(\overline{x}_1(\epsilon), \widehat{x}_2) + (1-p)\widetilde{\mathrm{P}}(x_1^{**}, \widehat{x}_2) - \mathrm{P}(x_1^{**}, \widehat{x}_2(x_1^{**}))] > p[\mathrm{C}_1(\overline{x}_1(\epsilon)) - \mathrm{C}_1(x_1^{**})].$$

This last inequality can always be met by choosing p and $\overline{x}_1(\epsilon)$ appropriately. To see this, it suffices to note that Eq. (5) holds for any $p \in (0,1)$ while the RHS of the above inequality goes to zero when $p \to 0$. Therefore, we can always find a $0 < p \leqslant \overline{p}$ that verifies the above inequality. The proof is then completed by using an argument of continuity. Since $\lim_{\epsilon \to 0} \sigma_1^\epsilon = \delta_{x_1^{**}}$, we have that $\lim_{\epsilon \to 0} \widetilde{\pi}_1(\sigma_1^\epsilon, \widehat{x}_2) = \pi_1(x_1^{**}, \widehat{x}_2(x_1^{**}))$. Moreover, the assumption $f_1(x_1^*) > f_2(x_2^*)$ implies that $\pi_1(x_1^{**}, x_2(x_1^{**})) > \pi_1(x_1^*, x_2^*)$. Hence, it suffices to consider the special case, μ_2^o with $\mu_2^o = \delta_{x_1}$ whenever $x_1 \in \mathrm{supp}(\sigma_1^\epsilon)$ is the actual choice of player 1 to conclude that there exists a mixed strategy σ_1^ϵ that strictly dominates x_1^{**}. Since, any other distribution μ_2^o implies that players obtain the payoffs of the simultaneous contest, we have that, $\pi_1(x_1^*, \widehat{x}_2(x_1^*)) > \pi_1(x_1, \widehat{x}_2(x_1))$, $\forall x_1 \neq x_1^*$, and hence an immediate application of Claim 0 proves that any pure action x_1 is strongly dominated by some mixture in Γ^δ when we consider the mixed extension of this game. This proves Lemma 0 in the main text. It remains to prove that the indifference condition holds. To do so, define $\Lambda : [0, \overline{\epsilon}] \to \mathbb{R}$ as

$$\Lambda(\epsilon) = \pi_1(\overline{x}_1(\epsilon), \widehat{x}_2(\mathrm{E}_{\beta_2^\tau(\cdot|\overline{x}_1(\epsilon))} \widetilde{x}_1)) - \pi_1(x_1^{**}, \widehat{x}_2(\mathrm{E}_{\beta_2^\tau(\cdot|x_1^{**})} \widetilde{x}_1)).$$

The concavity of payoffs imply that when we have a sufficiently convex function C_1 with $\lim_{x_1 \to \infty} \mathrm{C}_1(x_1) = \infty$ (together with the assumption that V_1 is bounded), we have that $\Lambda(\epsilon) > 0$ for some sufficiently small ϵ and $\Lambda(\epsilon) < 0$ for some large enough ϵ. Also, notice that for a support $\{x_1^{**}, \overline{x}_1(\epsilon)\}$, player 2's best reply mapping, $\widehat{x}_2(\mathrm{E}_{\beta_2^\tau(\cdot|\overline{x}_1(\epsilon))})$, is continuous in ϵ. Hence, by continuity of Λ we can apply the mean value Theorem to conclude the existence of a $\overline{\epsilon} > \epsilon^* > 0$ such that $\Lambda(\epsilon^*) = 0$. This condition is equivalent to the indifference condition for player 1 when he uses a mixed strategy σ_1^ϵ with support $\mathrm{supp}(\sigma_1^\epsilon) = \{x_1^{**}, x_1^{**} + \epsilon^*\}$. This completes the proof. ■

A.3 Signaling technologies for continuous mixed strategies probability distributions.

Depending on the finiteness of the support of the mixed strategy used by player 1, σ_1, Eq.(3) of the main text writes in general as,

$$\Pr{}^\tau\{\phi = x_1 \mid x_1\} = \tau + (1-\tau)\mu_2(x_1)$$

with,

$$\mu_2(x_1) \equiv \begin{cases} \sigma_1(x_1) \text{ if } |\mathrm{supp}(\sigma_1)| < \infty; \\ \Pr(a_1 \in [x_1, x_1 + dx_1]) \text{ otherwise,} \end{cases}$$

and where $|\mathrm{supp}(\sigma_1)|$ denotes the cardinality of the support of σ_1.

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